"The Strip Code and the Jetting of Gas Between Plates," by John G. Trulio.

"Cel: A Time-Dependent, Two-Space-Dimensional, Coupled Eulerian-Lagrange Code," by W. F. Noh.

"The Tensor Code," by G. Maenchen and S. Sack.

"Calculation of Elastic-Plastic Flow," by Mark L. Wilkins.

"Solution by Characteristics of the Equations of One-Dimensional Unsteady Flow," by N. E. Hoskin.

"The Solution of Two-Dimensional Hydrodynamic Equations by the Method of Characteristics," by D. J. Richardson.

"The Particle-In-Cell Computing Method for Fluid Dynamics," by Francis H. Harlow.

"The Time Dependent Flow of an Incompressible Viscous Fluid," by Jacob Fromm.

The editors have gathered together a variety of different methods and are to be congratulated for making them readily available in this volume. Some of the papers go into more detail than others but the general impression of all of them is a good one. Some examples are given, and the editors state that the next volume of the series will be devoted to hydrodynamics from an applied point of view.

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38[X].—N. K. BARY, *A Treatise on Trigonometric Series*, Pergamon Press, New York, 1965, Vol. I, xxiii + 553 pp., price \$12.00; Vol. II, xix + 508 pp., price \$15.00.

This is an interesting and valuable treatise. It contains a lot of material not otherwise available in book form. Moreover, following an old and well established Russian pedagogical tradition, the author pays much attention to the matter of presentation. The proofs are laid out carefully and in great detail, both mathematically and graphically, so that the difficulties of reading and following the arguments are at a minimum. Another important aspect of value to the beginner: the basic facts from various parts of the theory are assembled together in Chapter I of the book, which, preceded by an Introduction containing auxiliary material, may be treated as a textbook within the textbook, giving fundamentals without overwhelming the reader with excessive details. The book contains a considerable number of examples and exercises, often with hints of solutions.

The theory of trigonometric series has a long history. It has had a strong impact on other branches of mathematical analysis and has, in turn, been affected by developments in other fields. The modern theory emerged roughly some sixty years ago, primarily in connection with the appearance of Lebesgue's integral, but now we see a number of trends in the theory. These trends are not pure and are constantly influencing one another, but still they are easily discernible.

The oldest of them goes back to Lebesgue himself and treats the subject primarily as a branch of Real Variable. Problems are stated in terms of properties of real functions, and progress is achieved through a refined analysis of sets and functions. Of considerable importance here was the work of the Russian school. It could be traced back to Egorov, but *spiritus movens* here was actually Lusin. His M.A. thesis ("The integral and trigonometric series" in Russian, Moscow, 1915) as well as his personal influence made a very strong impact on Russian Mathematics and left trends which remain in existence even now, fifty years later. Lusin had very outstanding students, among them Khintchin, Menshov, Kolmogorov, and the author of this book—Nina Bary (she dedicates the book to "my teacher," Lusin). Of comparable significance for trigonometric series in the context of Real Variables was the work of Denjoy in France, though he was concerned with a somewhat different problematics and, unfortunately, did not have pupils to give further impetus to his ideas.

The second direction in trigonometric series recognizes the affinity of the subject with analytic functions and does not hesitate to borrow ideas and methods from the latter. If one wanted to label this direction, the names that come most naturally to one's mind would be those of Hardy and Littlewood, whose systematic work over the period of more than 30 years was probably the strongest individual influence on the field. If one wanted to mention others, the long list should begin with the names of Frederick and Marcel Riesz and those of Lusin and Privalov. The highly original work of the two Russians was an application of delicate methods of Real Variable to the study of boundary behavior of analytic functions and thence the behavior of trigonometric series; it remains a prototype of quite a lot of research done currently in similar but more general contexts.

In recent years a new trend is appearing in trigonometric series and is rapidly gaining strength—in the direction of Functional Analysis. The initial attempts to apply Functional Analysis to trigonometric series may be associated with the appearance of Banach's "Opérations Linéaires" (or even with some earlier work of F. Riesz), but the first major successes here are due to Beurling. It seems to be beyond doubt that this direction offers great possibilities, but only if amalgamated with methods of Real and Complex Variables. In spite of beliefs of many enthusiasts, it seems unlikely that Functional Analysis can tackle single-handedly the really difficult and significant problems of trigonometric series.

In addition to the three trends just described, one might mention another possibility, which lies almost totally in the future: tying trigonometric series with the theory of numbers. The very form of a trigonometric series—a superposition of harmonic oscillations in a fixed order—would indicate that arithmetic properties of real numbers could become a decisive tool, but the difficulty is that at present we cannot even ask intelligent questions here. The few results we know are very interesting, some of them even exciting, but they are too disconnected to offer any general hints.

But let us return to the book. Though the author tries to give a very complete presentation of the theory, it is clear that the methods of Real Variable are closest to her heart and, in view of what has been said above, this is not surprising. These methods are presented with mastery and very exhaustively. A specialist will be particularly appreciative of the chapters giving the results of Menshov concerning "adjustments" of functions in sets of small measure and representation by trigonometric series of general measurable functions. The original papers of Menshov were written in Russian and are not as well known as they deserve to be.

In what follows we give a brief description of the material covered by the book, warning the reader that this description cannot give a completely adequate idea of the contents.

The Introduction has the following sections: I. Analytical theorems (summation by parts, Second Mean-Value Theorem, Convex functions and sequences); II. Numerical series (in particular the methods (C, 1) and Abel's); III. Inequalities (Hölder, Minkowski, etc.); IV. Theory of sets and functions (relevant results from Lebesgue and Riemann-Stieltjes integral); V. Functional Analysis (linear functionals, spaces L^p); VI. Approximation of functions by trigonometric polynomials (best approximation, modulus of continuity, smoothness, Bernstein's inequality).

Chapter I is quite long (pp. 43–204) and gives a self-contained presentation of elements of the theory. It is roughly comparable with the Hardy-Rogosinski Cambridge Tract on Fourier series and contains both Fourier and Riemann theory. The main points are as follows: generalities about trigonometric series and Fourier series; the Riesz-Fischer theorem; series with monotonically decreasing coefficients; elementary tests for convergence of Fourier series; theorems of Fejér and Fejér-Lebesgue; Poisson's integral; factors of convergence; divergence and non-uniform convergence of Fourier series of continuous functions; the theorem of Lusin-Denjoy on absolute convergence; elements of Riemann's theory of trigonometric series (Cantor's lemma, Riemann's method of summation, formal multiplication, elementary theorems on uniqueness).

Chapter II gives a detailed discussion of Fourier coefficients: order of magnitude of coefficients of functions of bounded variation, of Lip k, of L^p (Hausdorff-Young); applications of Rademacher functions; Helson's theorem (trigonometric series with positive partial sums have coefficients tending to 0).

Chapter III deals with the convergence of a Fourier series at a point and gives all the classical tests not presented in Chapter I. Chapter IV is devoted to Fourier series of continuous functions: for uniform convergence we have the classical Dini-Lipschitz test and its generalization to an integral form (Salem); for divergence equally classical constructions (divergence in a prescribed denumerable set, in a dense set of the second category, uniform boundedness together with divergence in a set everywhere of the power of the continuum).

Chapter V deals with convergence and divergence of a Fourier series in sets. We have here the theorem of Kolmogorov-Seliverstov-Plessner for f in L^2; its extension by Marcinkiewicz (to f in L^p, $1 \leqq p \leqq 2$); tests of Salem and Marcinkiewicz and the proof that the latter cannot be strengthened; convergence of Fourier series and capacities of sets; Kolmogorov's construction of an everywhere divergent Fourier series; Ulyanov's theorem that if f is not in L^2, then its Fourier series can be so rearranged that the resulting series diverges almost everywhere.

Chapter VI on "Adjustments" of functions in sets of small measure is devoted almost exclusively to results of Menshov. Two main results are, first, that any integrable function can be so modified in a set of arbitrarily small measure that the resulting function has a uniformly convergent Fourier series and, second, that, given any closed non-dense set P, we can modify f outside P so that the new Fourier series converges almost everywhere.

This is the final chapter of Vol. I of the book, and is followed by a number of addenda dealing with a number of auxiliary but advanced topics from Functional Analysis and Complex Variables (the Banach-Steinhaus theorem, the Phragmén-Lindelöf Principle, etc.).

Chapter VII, Vol. II deals with various kinds of summabilities of numerical series: $(C^*, 0)$, Lebesgue, Rogosinski (usually called Bernstein-Rogosinski in Russian literature) and $(C^*, 0)$ (the latter bears the same relation to ordinary convergence as non-tangential approach to radial approach in Abel's summability). The main results of the chapter are proofs of the Hardy-Littlewood theorem about strong summability of Fourier series and its extension from L^p, $p > 1$, to L^1 (due to Marcinkiewicz).

Chapter VIII, "Conjugate trigonometric series," is an introduction to complex methods. It mainly centers around M. Riesz's theorem (that a function conjugate to a function in L^p, $p > 1$, is also in L^p) and its various extensions and complements, but also gives the proof of the fundamental theorem of Plessner asserting that if a trigonometric series converges in some set, the conjugate series converges almost everywhere in the set. A section is also devoted to a description of a new kind of integral (the A-integral) generalizing that of Lebesgue, and gives a few applications of the notion to Fourier series.

Chapter IX discusses the absolute convergence of Fourier series, and in particular proves the Wiener-Lévy theorem. To the specialist, however, this chapter will be of interest not primarily because of the fundamental results, which are very well known, but because it brings to light a number of connections between absolute convergence and the various notions of best approximation. These theorems are not as well known as they deserve to be, and it is perhaps a pity that in some interesting cases the results are stated without proof.

In Chapter X we have a study of properties of sine and cosine series with monotone coefficients and, in particular, the asymptotic behavior (due to Salem) of such series near the origin.

The whole of Chapter XI is devoted to lacunary series. Such series are quite special but interesting for several reasons. In the first place, their behavior resembles that of series of independent random variables in the calculus of probabilities; second, they have a number of sharp properties as regards, say, convergence almost everywhere, uniform convergence, uniqueness, best approximation, etc. Finally, and this is the most important, they are a rich source of examples illuminating various points of the general theory.

The next two chapters deal with convergence, ordinary and absolute, of general trigonometric series. They contain very many results which are rather disconnected, but this is just the picture of the situation as it exists at present, and which is likely to persist until some major breakthrough is achieved. In any case, this is a good source of information.

The remaining two chapters are probably the most interesting in this volume. The first deals with the theory of uniqueness in which the author herself was very much interested and obtained some fundamental results. It begins with the classical theorems of du Bois Reymond, W. H. Young, and de la Vallée-Poussin, goes through theorems of Menshov, Rajchman and her own (the first examples of perfect sets of uniqueness and multiplicity of measure zero) and discusses the most modern developments, the most important of which are due to Salem (and indicate unexpected links between trigonometric series and algebraic numbers). The chapter gives a very exhaustive presentation of the situation as it exists now.

The last chapter is interesting for a different reason: it gives detailed proofs of a number of theorems little known outside Russia concerning the representation of an arbitrary measurable function by an almost everywhere convergent trigonometric series. That this is possible was first shown by Menshov some twenty years ago. Nina Bary completed the result by showing that the series can be obtained by termwise differentiation of the Fourier series of a continuous function. Of course, Menshov's series is not unique, and the problem whether the function can also take the value $+\infty$ in a set of positive measure still remains open (it is conjectured that no trigonometric series can diverge to $+\infty$ in a set of positive measure).

Volume II also terminates with a long list of appendices.

The English translation is graphically attractive but, unfortunately, it has a number of defects. Obvious mathematical misprints of the Russian original have been retained in the translation, and a number of serious distortions introduced. (To give examples from Chapter I: the expression "in the metric of the space L^2" of the original is systematically translated "in the metric space L^2"; in Steinhaus' theorem on p. 104 the word "equiconvergent" is replaced by "convergent" making the formulation incomprehensible; at the bottom of p. 110 we find the following passus: ". . . in fact, it can be proved that for a bounded function the partial sums of a Fourier series should be bounded. However, this is untrue even for continuous functions.") Such distortions are no real obstacle to a specialist, but may prove serious stumbling blocks to the beginner. Finally, one more point which, however, applies to many translations from Russian: references to Russian textbooks give pages of the original, even if the textbook has meanwhile been translated into English. Such references are useless for readers not familiar with the Russian language.

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39[X].—MARTIN BECKER, *The Principles and Applications of Variational Methods*, The M.I.T. Press, Cambridge, Massachusetts, 1964, vii + 120 pp., 24 cm. Price $5.00.

The present book is concerned with variational methods for solving problems in science and engineering. Various methods are described in detail, giving the advantages and disadvantages of each method. The author restricts himself mainly to "trial function methods" for solving an equation of the form $H\varphi = f$, where H is a given operator and f is a known function. In these methods a set of trial functions $\varphi_1, \cdots, \varphi_N$ are given. The problem is to determine a linear combination $a_1\varphi_1 + \cdots + a_N\varphi_N$ which is a "best" approximation to the solution φ of $H\varphi = f$. Particular attention is given to the method of least squares.

The book begins with the study of self-sufficient equations, that is, equations that are the Euler equations of a functional J. The method of adjoint functions is developed for non-self-sufficient problems. The method of weighted residuals is discussed as well as various methods for solving equations subject to constraints.

The author sets up a set of desirable criteria for variational methods and shows